The research of the optimal control problem by the systems with maximum

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Introduction

Delay differential equations are a type of differential equation in which the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times.

$$\dot{x}(t) = f(t, x(t), x(t - \tau)), \quad \tau > 0.$$

Firstly, this kind of differential equation appeared in literature of the late XVIII century (M. de Condorcet, 1771, " Memoires de Academie des Sciences", L.Euler, J. Bernoulli).

Differential equations with maximum is one special type of functional differential equations is the case when the evolutionary equations use the maximum of the studied function on a past time interval

$$\dot{x} = f(t, x(t), \max_{s \in [t-h,t]} x(s)),$$

where $x(t) \in \mathbb{R}^n$ is a phase vector; $t \in [t_0, t_1]$ is time of the system existence; $f : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is n - dimensional vector function;
$\max_{s \in [t-h,t]} x(s) = (\max_{s \in [t-h,t]} (x_1(s)), ..., \max_{s \in [t-h,t]} (x_n(s)))$.
The maximum of the unknown function $x(t)$ could be given

1. on an interval with fixed length, i.e., $\max_{s \in [t-r,t]} x(s), r = const > 0$;
2. on an interval with variable length, i.e., $\max_{s \in [g(t),\gamma(t)]} x(s)$, where $t_0 \leq g(t) \leq \gamma(t) \leq t$;
3. on several different intervals with fixed lengths or variable lengths.

The study of equations which includes a maximum of the unknown function is spread to various types differential equations. We will mention some of them:

1. some properties of the solutions of differential equations with maximum are studied by A. R. Magomedov, Y. Ryabov, G.Nabiev;
2. oscillatory properties D.D.Bainov, A.Zahariev, D.Kolev, N. Markova ;
3. approximate solutions D.D.Bainov and S.Hristova;
4. averaging O. D. Kichmarenko and V. A. Plotnikov, D.D.Bainov, V.P Shpakovih, V.I.Muntyan;

1. Differential equation with maximum

1.1. The existence and uniqueness of the Cauchy problem solution with maximum

$$\dot{x}(t) = f(t, x(t), \max_{s\in[g(t),\gamma(t)]} x(s)), \quad x(t_0) = x_0, \qquad (1)$$

here $x(t) \in \mathbb{R}^n$ is a phase vector; $t \in [t_0, t_1]$ is time of the system existence; $f : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$; $g(t)$ and $\gamma(t)$ are known functions, and $t_0 \le g(t) \le \gamma(t) \le t$; $\max_{s\in[g(t),\gamma(t)]} x(s) = (\max_{s\in[g(t),\gamma(t)]}(x_1(s)), ..., \max_{s\in[g(t),\gamma(t)]}(x_n(s)))$.
Continuous function $x : [t_0, t_1] \to \mathbb{R}^n$ will be considered as a solution of problem (1).
Note that if $g(t) = \gamma(t) = t - h$, then (1) is a differential equation with constant delay and if $g(t) = \gamma(t)$, then (1) is a differential equation with variable delay.

Theorem

Let the function $f(t, x, y)$ is continuous by set of arguments in the neighborhood of the point (t_0, x_0, x_0) and exists Lipschitz constant L i.e.

$$\|f(t, x, y) - f(t, x', y')\| \le L \left[\|x - x'\| + \|y - y'\| \right].$$

Then the solution of the Cauchy problem exists and it is uniqueness in some neighborhood of the point (t_0, x_0, x_0).

1.2 Averaging method for differential equations with maximum

The Cauchy problem

$$\dot{x}(t) = \varepsilon f(t, x(t), \max_{s \in [g(t), \gamma(t)]} x(s)), \quad x(t_0) = x_0 \tag{2}$$

with maximum and small parameter $\varepsilon > 0$, $t > 0$ is considered.
Let us consider the following averaged equation

$$\dot{y}(t) = \varepsilon f^0 \left(y(t), \max_{s \in [g(t), \gamma(t)]} y(s) \right), \quad y(t_0) = x_0 \tag{3}$$

for the equation (2). Here

$$f^0(x, y) = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(t, x, y) dt. \tag{4}$$

Note that system (3) is an autonomous system due to right part of equation does not depend on t.

Theorem

Let in $Q = [t \geq 0, x, y \in D \subset \mathbb{R}^n]$the following conditions hold:
1) $f(t, x, y)$ is a continuous function on t and

$$\|f(t, x, y)\| \leq M, \quad (5)$$

$$\|f(t, x, y) - f(t, \bar{x}, \bar{y})\| \leq \lambda[\|x - \bar{x}\| + \|y - \bar{y}\|], \quad (6)$$

2) $g(t)$ and $\gamma(t)$ are evenly continuous functions and $0 \leq g(t) \leq \gamma(t) \leq t$;
3) the limit (4) exists evenly with respect to x, y;
4) the solution of the equation (3) and condition $y(0) \in D^{''} \subset D$ together with its $\rho-$neighbourhood belongs to D.
Then for any $\eta > 0$, $L > 0$ there exists $\varepsilon^0(\eta, L)$ such that for any $\varepsilon \in (0, \varepsilon^0]$ the following estimate holds:

$$\|x(t) - y(t)\| \leq \eta, \quad (7)$$

where $x(t)$, $y(t)$ are solutions of systems (2) and (3) accordingly, $x(0) = y(0) \in D^{''}$.

1.3 Example

$$\begin{cases} \dot{x_1}(t) = \varepsilon \left[-1.4x_1 \sin\left(t + \max_{s\in[g(t),\gamma(t)]} x_2(s) \right) + 0.2x_1^3 \cos^3(t+x_2) \right] \sin(t+x_2), \\ \dot{x_2}(t) = -\frac{\varepsilon}{x_1} \left[-1.4x_1 \sin\left(t + \max_{s\in[g(t),\gamma(t)]} x_2(s) \right) + 0.2x_1^3 \cos^3(t+x_2) \right] \cos(t+x_2), \end{cases} \tag{8}$$

$$x_1(0) = 2, \quad x_2(0) = \frac{\pi}{2},$$

$g(t) = \max\left\{0, t - \frac{1}{2}\right\}$, $\gamma(t) = \max\left\{0, t - \frac{1}{4}\right\}$.

Averaged system

$$\begin{cases} \dot{y_1}(t) = \varepsilon \left[-1.4y_1 \cos\left(t + \max_{s\in[g(t),\gamma(t)]} y_2(s) \right) \right], \\ \dot{y_2}(t) = -\frac{\varepsilon}{y_1} \left[-1.4y_1 \sin\left(t + \max_{s\in[g(t),\gamma(t)]} y_2(s) \right) + \frac{3}{8} 0.2y_1^3 \right]. \end{cases} \tag{9}$$

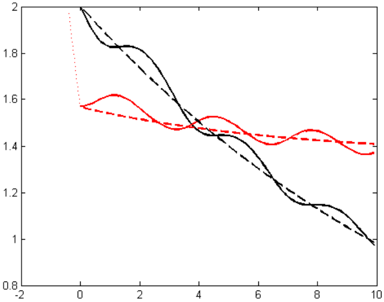

Figure 1: $\varepsilon = 0.1$

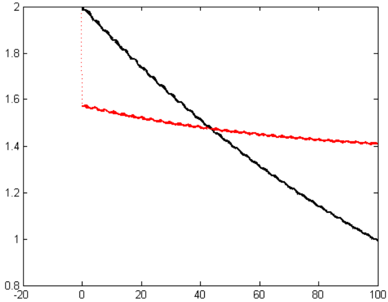

Figure 2: $\varepsilon = 0.01$

In the table calculations of values $\max|x_1(t) - y_1(t)|$, $\max|x_2(t) - y_2(t)|$, and $\|x(t) - y(t)\|$ are presented.

ε	$\max\|x_1(t) - y_1(t)\|$	$\max\|x_2(t) - y_2(t)\|$	$\max\\|x(t) - y(t)\\|$
0.5	0.4935	0.6312	0.6425
0.1	0.0909	0.0903	0.0996
0.01	0.0092	0.0085	0.0100
0.001	0.00092	0.00085	0.0010

O.D.Kichmarenko, K.Yu.Sapozhnikova,
Full averaging scheme for differential equation with maximum,
Contemporary Analysis and Applied Mathematics,
Istanbul, Vol.3, No1, pp. 113-122, 2015.

2. Optimal control problem with maximum of state

$$\dot{x} = f(t, x(t), \max_{s\in[g(t),\gamma(t)]} x(s), u(t)), \qquad x(0) = x_0. \tag{10}$$

Here $x(t) \in \mathbb{R}^n$ is a phase vector; $t \in [0, t_1]$ is time of the system existence; $g(t)$ and $\gamma(t)$ are known functions, and $0 \le g(t) \le \gamma(t) \le t$;
$\max_{s\in[g(t),\gamma(t)]} x(s) = (\max_{s\in[g(t),\gamma(t)]}(x_1(s)), ..., \max_{s\in[g(t),\gamma(t)]}(x_n(s)))$;
$f : [0, t_1] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$; $u(t)$ is a control function and $u(t) \in U \subset comp(R^r)$.

The solution of the problem (10) is an absolutely continuous function with fixed control functions $u(t)$.
The admissible controls are piecewise continuous functions $u(t) : [0, t_1] \to U$.
Denote set of all admissible functions as $\mathfrak{U}$.

The quality of the control system (10) is evaluated by the functional

$$J[u] = \int_0^{t_1} \mu(x(t), u(t))\, dt + \varphi(x(t_1)). \quad (11)$$

The problem (10)-(11) is called Bolz problem (optimal control problem with unfixed right trajectory end).
The object is to find the optimal control $u^0(t)$ and the corresponding trajectory $x^0(t)$, $t \in [0, t_1]$ which minimize the functional (11), i.e.

$$J[u^0] = \int_0^{t_1} \mu(x^0(t), u^0(t))\, dt + \varphi(x^0(t_1)) = \min_{u \in \mathfrak{U}} J[u].$$

2.1. Averaging scheme for optimal control problem with maximum of state

$$\dot{x}(t) = \varepsilon \left[f(t, x(t), \max_{s \in [g(t), \gamma(t)]} x(s)) + A(x(t), \max_{s \in [g(t), \gamma(t)]} x(s)) \varphi(t, u) \right],$$

$$x(0) = x_0, \qquad (12)$$

where $x(t) \in \mathbb{R}^n$ is a phase vector; $f : [0, L\varepsilon^{-1}] \times \mathbb{R}^n \times \mathbb{R}^n \times \to \mathbb{R}^n$; $\varepsilon > 0-$ small parameter; $t \in [0, L\varepsilon^{-1}]$ is time of the system existence; $g(t)$ and $\gamma(t)$ are known functions, and $0 \leq g(t) \leq \gamma(t) \leq t$; $A - n \times m$ - matrix; $\varphi : [0, L\varepsilon^{-1}] \times U \to \mathbb{R}^m$ - vector function; $\max_{s \in [g(t), \gamma(t)]} x(s) = (\max_{s \in [g(t), \gamma(t)]} (x_1(s)), ..., \max_{s \in [g(t), \gamma(t)]} (x_n(s)))$; $u(t)$ is a control function $u(t) \in U \subset comp(R^r)$, $u \in \mathfrak{U}$.

Averaged system is defined by:

$$\dot{y}(t) = \varepsilon \left[\bar{f}(y(t), \max_{s\in[g(t),\gamma(t)]} y(s)) + A(y(t), \max_{s\in[g(t),\gamma(t)]} y(s))v(t) \right],$$

$$y(0) = x_0, \tag{13}$$

where

$$\bar{f}(x, y) = \frac{1}{2\pi} \int_0^{2\pi} f(t, x, y)dt,$$

$v(t) \in V$ new control vector, here

$$V = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \varphi(t, u(t))dt, u(t) \in U \right\}, \quad V \in comp(R)^m. \tag{14}$$

Denote set of all admissible functions as $\mathfrak{V}$.

Algorithm 1.

Establish the correspondence between control functions u(t) and v(t)

1. For control $v \in \mathfrak{V}$ find the correspondence control $u \in \mathfrak{U}$ in the following way:
   1. calculate points $v_i = \frac{1}{2\pi} \int\limits_{2\pi i}^{2\pi(i+1)} v(t)dt$,
   2. assign control $u(t) = \{u_i(t), 2\pi i \le t \le 2\pi(i+1), i = 1, 2, ..\}$, where $u_i(t)$ can be obtained from the conditions:

$$\underset{u(t)\in U}{\operatorname{argmin}} \left\| \frac{1}{2\pi} \int\limits_{2\pi i}^{2\pi(i+1)} \varphi(t, u(t))dt - v_i \right\|. \qquad (15)$$

2. For control $u \in \mathfrak{U}$ find the correspondence control $v \in \mathfrak{V}$ in the following way:
   1. denote values $u_i = \{u(t), 2\pi i \le t \le 2\pi(i+1), i = 1, 2, ..\}$ and calculate points $w_i = \frac{1}{2\pi} \int\limits_{2\pi i}^{2\pi(i+1)} \varphi(t, u_i(t))dt$;
   2. assign control $v(t) = \{v_i(t), 2\pi i \le t \le 2\pi(i+1), i = 1, 2, ..\}$, where v_i can be obtained from condition: $\underset{v\in V}{\operatorname{argmin}} \|w_i - v\|$.

Theorem

Suppose that in domain $Q = \{t \geq 0, x, y \in D \subset \mathbb{R}^n, u \in U \subset comp(\mathbb{R}^r)\}$ the following conditions hold:

1) $f(t, x, y)$ is continuous function on t, uniform bounded by constant M and satisfy Lipschitz condition with respect x, y;
2) $\varphi(t, u)$ is continuous function on t, u,$2\pi -$ periodic on t and uniform bounded by constant M;
3) $A(x, y)$ is continuous function, $a_{ij}(x, y)$ are continuous functions on x, y $(i = \overline{1, n}, j = \overline{1, m})$ and $\|A(x, y)\| \leq M$;
4) $g(t)$, $\gamma(t)$ are evenly continuous functions;
5) for any admissible control $v \in \mathfrak{V}$ the corresponding trajectory $y(t)$ of the average system (13), $y(0) = x_0 \in D'$ is defined by $t \geq 0$, and with its $\rho -$ neighborhood is in domain D.

Then there exists $\varepsilon^0 > 0, \quad C > 0$ such that for any $\varepsilon \in (0, \varepsilon^0]$ and $t \in [0, L\varepsilon^{-1}]$ following statements hold:

1) for any admissible control $u \in \mathfrak{U}$ of system (12) exists control $v \in \mathfrak{V}$ of system (13) such that:

$$\|x(t) - y(t)\| \leq \varepsilon C, \tag{16}$$

   where $x(t)$ is a solution of system (12) corresponding to control u and $y(t)$ is a solution of system (13) which corresponds to control v and $y(0) = x_0 \in D' \subset D$;
2) for any admissible control $v \in \mathfrak{V}$ of system (13) exists control $u \in \mathfrak{U}$ of system (12) such that (16) is satisfied.

2.2 Example

$$\begin{cases} \dot{x_1}(t) = \varepsilon \left[-1.4x_1 \sin\left(t + \max\limits_{s\in[g(t),\gamma(t)]} x_2(s) \right) + 0.2x_1^3 \cos^3(t + x_2) + u \right] \sin(t + x_2), \\ \dot{x_2}(t) = -\frac{\varepsilon}{x_1} \left[-1.4x_1 \sin\left(t + \max\limits_{s\in[g(t),\gamma(t)]} x_2(s) \right) + 0.2x_1^3 \cos^3(t + x_2) + u \right] \cos(t + x_2), \end{cases} \tag{17}$$

$$x_1(0) = 2, \quad x_2(0) = \frac{\pi}{2},$$

$g(t) = \max\left\{0, t - \frac{1}{2}\right\}$, $\gamma(t) = \max\left\{0, t - \frac{1}{4}\right\}$, $\varepsilon > 0$ small parameter, $u \in U$.

$$\begin{cases} \dot{y_1}(t) = \varepsilon \left[-1.4y_1 \cos\left(t + \max\limits_{s\in[g(t),\gamma(t)]} y_2(s) + \cos\ y_2 v_1 \right) \right], \\ \dot{y_2}(t) = -\frac{\varepsilon}{y_1} \left[-1.4y_1 \sin\left(t + \max\limits_{s\in[g(t),\gamma(t)]} y_2(s) \right) + \frac{3}{8} 0.2y_1^3 + \cos\ y_2 v_2 \right]. \end{cases} \tag{18}$$

Let control function of averaged system is known:

$$v = \begin{pmatrix} -\frac{1}{\pi^2}\sin\frac{t}{2} \\ \cos t \end{pmatrix},$$

According to the algorithm 1 we get corresponding control to initial system:

$$u1(t)=\left\{u_i^1(t)=\sin(-1.58966t+4.17932\pi i+3.17932\pi), t\in[2\pi i, 2\pi(i+1)], i=1,2,..\right\}$$

$$u2(t)=\left\{u_i^2(t)=\sin(0.6487t+0.2974\pi i+1.2974\pi), t\in[2\pi i, 2\pi(i+1)], i=1,2,..\right\},$$

$$u3(t)=\left\{u_i^3(t)=\sin(-0.6487t-3.2974\pi i-1.2974\pi), t\in[2\pi i, 2\pi(i+1)], i=1,2,..\right\}.$$

Then we research control functions u1, u2, u3

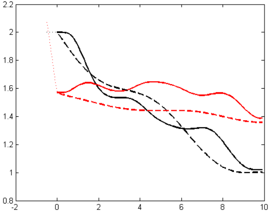

Figure 1: u1, $\varepsilon = 0.1$

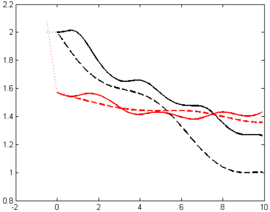

Figure 2: u2, $\varepsilon = 0.1$

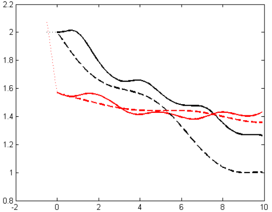

Figure 3: u3, $\varepsilon = 0.1$

For u1 we research proximity of trajectories

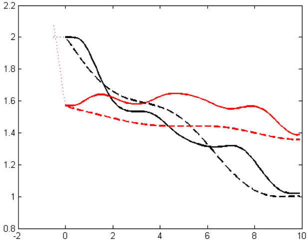

Figure 1: $\varepsilon = 0.1$

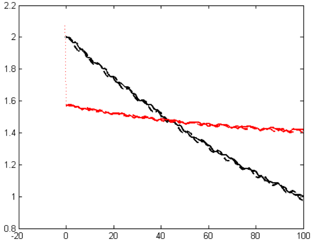

Figure 2: $\varepsilon = 0.01$

In the table calculations of values $\max|x_1(t) - y_1(t)|$, $\max|x_2(t) - y_2(t)|$, and $\|x(t) - y(t)\|$ are presented.

ε	$\max\|x_1(t) - y_1(t)\|$	$\max\|x_2(t) - y_2(t)\|$	$\max\|\|x(t) - y(t)\|\|$
0.5	0.4917	0.3796	0.4945
0.1	0.2071	0.2057	0.2573
0.01	0.0377	0.0198	0.0377
0.001	0.0100	0.0152	0.0101

Optimal control problem with maximum of control function

3.1 Pontryagin principle maximum

$$\dot{x} = f(t, x(t), u(t), \max_{s \in [g(t), \gamma(t)]} u(s)), \tag{19}$$

$$x(t_0) = x_0. \tag{20}$$

Here $x(t) \in (\mathbb{R}^n)$ is a phase vector, $t \in [t_0, t_1]$ is time of the system existence;
$u : [t_0, t_1] \to \mathbb{U}$ is a control function and $U \in \text{comp}(\mathbb{R}^r),\ u \in \mathfrak{U}$;
$f : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$; $g(t), \gamma(t)$ are known and evenly continuous functions, $t_0 \leq g(t) \leq \gamma(t) \leq t$;
$\max\limits_{s \in [g(t), \gamma(t)]} u(s) = (\max\limits_{s \in [g(t), \gamma(t)]}(u_1(s)), ..., \max\limits_{s \in [g(t), \gamma(t)]}(u_r(s)))$.

The object is to find optimal control $u^0(t)$, $t \in [t_0, t_1]$ and corresponding trajectory $x^0(t)$, $t \in [t_0, t_1]$ which minimize the functional

$$J[u] = \varphi(x(t_1)) \to \min \tag{21}$$

Then, defined sets M as:

$$M^{++} = \left\{ \bigcup_{i=1}^{k^{++}} [\alpha_i^{++}, \beta_i^{++}), \dot{g}(t) > 0, \dot{\gamma}(t) > 0, t \in [\alpha_i^{++}, \beta_i^{++}) \right\},$$

$$M^{+-} = \left\{ \bigcup_{i=1}^{k^{+-}} [\alpha_i^{+-}, \beta_i^{+-}), \dot{g}(t) > 0, \dot{\gamma}(t) < 0, t \in [\alpha_i^{+-}, \beta_i^{+-}) \right\},$$

...

$$M^{00} = \left\{ \bigcup_{i=1}^{k^{00}} [\alpha_i^{00}, \beta_i^{00}), \dot{g}(t) = \mathrm{const}, \dot{\gamma}(t) = \mathrm{const}, t \in [\alpha_i^{00}, \beta_i^{00}) \right\}.$$

The increasing functions $g^+(t) \equiv g(t)$ and $\gamma^+(t) \equiv \gamma(t)$, $t \in [\alpha^{++}, \beta^{++})$ have opposite increasing functions $\xi_{g^+(t)}(t) = \xi_{g^+}(t)$ and $\xi_{\gamma^+(t)}(t) = \xi_{\gamma^+}(t)$, $t \in [\alpha^{++}, \beta^{++})$. At the same time, decreasing functions $g^-(t) \equiv g(t)$ and $\gamma^-(t) \equiv \gamma(t)$,$t \in [\alpha^{--}, \beta^{--})$ have opposite decreasing functions $\xi_{g^-(t)}(t) = \xi_{g^-}(t)$,$\xi_{\gamma^-(t)}(t) = \xi_{\gamma^-}(t)$, $t \in [\alpha^{--}, \beta^{--})$ respectively.

We consider subsets:

$$\omega = \{\Theta \in [t_0, t_1] : \exists \tau \in M^*, g(\tau) = \Theta, \dot{g}(\tau) = 0, \gamma(\tau) = \Theta, \dot{\gamma}(\tau) = 0\},$$

$$\omega_0 = \left\{\Theta \in [t_0, t_1] : \exists i \in \overline{1, k^{00}}, g_i^0(\tau) = \Theta, \gamma_i^0(\tau) = \Theta, \tau \in \left[\alpha_i^{00}, \beta_i^{00}\right)\right\},$$

where $M^* = M^{++} \cup M^{+-} \cup M^{+0} \cup M^{-+} \cup M^{--} \cup M^{-0} \cup M^{0+} \cup M^{0-}$.

Let consider the case where $\theta \notin \omega_0 \cup \omega$. For θ we define the sets of indexes:

$$I^{++}(\theta) = \left\{ i \in \left\{ \overline{1, k^{++}} \right\} : \exists \tau \in [\alpha_i^{++}, \beta_i^{++}), g_i^{+}(\tau) = \theta, \gamma_i^{+}(\tau) = \theta \right\},$$

$$I^{+-}(\theta) = \left\{ i \in \left\{ \overline{1, k^{+-}} \right\} : \exists \tau \in [\alpha_i^{+-}, \beta_i^{+-}), g_i^{+}(\tau) = \theta, \gamma_i^{-}(\tau) = \theta \right\},$$

...

$$I^{0-}(\theta) = \left\{ i \in \left\{ \overline{1, k^{0-}} \right\} : \exists \tau \in [\alpha_i^{0-}, \beta_i^{0-}), g_i^{0}(\tau) = \theta, \gamma_i^{-}(\tau) = \theta \right\}.$$

The adjoint system is defined by:

$$\dot{\psi}(t) = -\frac{\partial H(x(t), \psi(t), u(t), \max\limits_{s\in[g(t),\gamma(t)]} u(s), t)}{\partial x}\psi;\;\; \psi(t_1) = -\frac{\partial \varphi(x(t_1))}{\partial x}, \qquad (22)$$

Defined the Hamiltonian function H as follows:

$$H(x(t), \psi(t), u(t), \max\limits_{s\in[g(t),\gamma(t)]} u(s), t) = \psi'(t) f(t, x(t), u(t), \max\limits_{s\in[g(t),\gamma(t)]} u(s).$$

$$\overline{\Delta} H_v(x(t), \psi(t), u(t), \max\limits_{s\in[g(t),\gamma(t)]} u(s), t) =$$
$$H(x(t), \psi(t), \max\limits_{s\in[g(t),\gamma(t)]} u(s), v(t), t) - H(x(t), \psi(t), u(t), \max\limits_{s\in[g(t),\gamma(t)]} u(s), t).$$

Theorem

Let $u(t), t \in [t_0, t_1]$ is an admissible control and $x(t), t \in [t_0, t_1]$ is a corresponding trajectory in task (19)-(20). In order that $u(t)$ and $x(t)$ be optimal it is necessary then there exist a nonzero solution of task (22) $\psi(t)$, $t \in [t_0, t_1]$ and the following condition holds:

$$\Lambda = \Delta_v H(t) +$$

$$+ \sum_{i \in I^{++}(t)} \left[\bar{\Delta}_v H(\xi_{g_i^+}(t)) \dot{\xi}_{g_i^+}(t) + \bar{\Delta}_v H(\xi_{\gamma_i^+}(t)) \dot{\xi}_{\gamma_i^+}(t) \right] +$$

...

$$+ \sum_{i \in I^{0-}(t)} \left[-\bar{\Delta}_v H(\xi_{\gamma_i^-}(t)) \dot{\xi}_{\gamma_i^-}(t) \right] \leq 0$$

$$\forall v \in U, t \in [t_o, t_1], \text{if } \ I^{++} \cup I^{+-} \cup I^{+0} \cup I^{-+} \cup I^{--} \cup I^{-0} \cup I^{0+} \cup I^{-0} \neq \varnothing;$$

$$\Delta_v H(t) \leq 0 \forall v \in U, t \in [t_0, t_1), \text{if } \ I^{++} \cup I^{+-} \cup I^{+0} \cup I^{-+} \cup I^{--} \cup I^{-0} \cup I^{0+} \cup I^{-0} = \varnothing$$

O.D.Kicmarenko, K.Yu.Sapozhnikova,
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3.2 Averaging scheme for optimal control problem with maximum of control function

$$\dot{x}(t) = \varepsilon \left(f(t, x(t)) + A(x(t))\varphi(t, u(t), \max_{s\in[g(t),\gamma(t)]} u(s)) \right), \quad x(0) = x_0, \qquad (23)$$

where $x(t) \in \mathbb{R}^n$ is a phase vector, $f : [0, L\varepsilon^{-1}] \times \mathbb{R}^n \to \mathbb{R}^n$ is 2π-periodic vector function on t; $\varepsilon > 0$ — small parameter; $t \in [0, L\varepsilon^{-1}]$ is a time of system exists; $A - n \times m-$ matrix, $a_{ij} : \mathbb{R}^n \to \mathbb{R}^1$; $\varphi : [0, L\varepsilon^{-1}] \times U \times U \to \mathbb{R}^m$, — 2π—periodic vector function on t; $g(t), \gamma(t)$ - are known functions and $0 \le g(t) \le \gamma(t) \le t$; u —is a control function, $u(t) \in U \subset comp(\mathbb{R}^r)$, $u \in \mathfrak{U}$, $\max_{s\in[g(t),\gamma(t)]} u(s) = (\max_{s\in[g(t),\gamma(t)]}(u_1(s)), ..., \max_{s\in[g(t),\gamma(t)]}(u_r(s)))$.

Averaged system is defined by:

$$\dot{y}(t) = \varepsilon\left[\overline{f}(y(t)) + A(y(t))v(t)\right], \quad y(0) = x_0, \tag{24}$$

where

$$\overline{f}(y(t)) = \frac{1}{2\pi}\int\limits_0^{2\pi} f(t, x(t))dt, \tag{25}$$

$v(t) \in V$ is a range of new control function:

$$V = \left\{ \frac{1}{2\pi}\int\limits_0^{2\pi} \varphi(t, u(t), \max_{s\in[g(t),\gamma(t)]} u(s))ds, u(t) \in U \right\}, \tag{26}$$

Algorithm 2

Establish the correspondence between control functions $u(t) \in U$ and $v(t) \in V$.

1. For control $v \in \mathfrak{V}$ find the correspondence control $u \in \mathfrak{U}$ in the following way:
   1. calculate points $v_i = \frac{1}{2\pi} \int\limits_{2\pi i}^{2\pi(i+1)} v(t)dt$,
   2. assign control $u(t) = \{u_i(t), 2\pi i \leq t \leq 2\pi(i+1), i = 1, 2, ..\}$, where $u_i(t)$ can be obtained from the conditions:

$$\arg \min_{u(t) \in U} \left\| \frac{1}{2\pi} \int\limits_{2\pi i}^{2\pi(i+1)} \varphi(t, u(t), \max_{s \in [g(t), \gamma(t)]} u(s))dt - v_i \right\|. \qquad (27)$$

2. For control $u \in \mathfrak{U}$ find the correspondence control $v \in \mathfrak{V}$ in the following way:
   1. denote values $u_i = \{u(t), 2\pi i \leq t \leq 2\pi(i+1), i = 1, 2, ..\}$ calculate points $w_i = \frac{1}{2\pi} \int\limits_{2\pi i}^{2\pi(i+1)} \varphi(t, u(t), \max_{s \in [g(t), \gamma(t)]} u(s))dt$;
   2. assign control $v(\varepsilon t) = \{v_i(t), 2\pi i \leq t \leq 2\pi(i+1), i = 1, 2, ..\}$, where v_i can be obtained from condition: $\arg \min_{v \in V} \|w_i - v\|$.

It is obvious that $u(t)$ and $v(t)$ can be determined ambiguously.

Theorem

Suppose that in domain $Q = \{t \geq 0, x, y \in D \subset \mathbb{R}^n, u \in U \subset comp(\mathbb{R}^r)\}$ the following conditions hold:

1) $f(t, x)$ is continuous function on t, $2\pi-$ periodic and bounded by constant K_1, satisfy the Lipschitz condition with respect to x and with constant λ_1;
2) matrix $A(x)$ is bounded by constant K_2 and satisfy Lipschitz condition with constant λ_2;
3) $\varphi(t, u)$ is $2\pi -$ periodic function and bounded by constant M;
4) $g(t)$, $\gamma(t)$ are evenly continuous function;
5) for any admissible control $v(t) \in V$ the corresponding trajectory $y(t)$ of the averaged system (23), $y(0) = x_0 \in D'$ is defined by $t \geq 0$, and with its $\rho -$ neighborhood is in domain D.

Then there exists $\varepsilon^0 > 0$, $C > 0$ such that for any $\varepsilon \in (0, \varepsilon^0]$ and $t \in [0, L\varepsilon^{-1}]$ following statements hold:

1) for any admissible control $u \in \mathfrak{U}$ system (23) exists control function $v \in \mathfrak{V}$ system (24), such that:

$$\|x(t) - y(t)\| \leq \varepsilon C, \tag{28}$$

where $x(t), y(t)$ are solutions of systems (23) and (24) accordingly, $x(0) = y(0) \in D' \subset D$.

2) $g(t)$ and $\gamma(t)$ evenly continuous functions;

3) for any admissible control $v \in \mathfrak{V}$ system (24) exists the control function $u \in \mathfrak{U}$ system (23), such that (28) is holds.

THANK YOU FOR YOUR ATTENTION